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On a General Class of Multivariate Linear Smoothing Operators

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1. INTRODUCTION

The purpose of this paper is to introduce and study a very general class of multivariate linear smoothing operators, which may not be positive, using both summation and integration. The integral portion is mainly used to smooth the data functions, which are only possibly measurable, while the summation portion is used to give good approximation from a simple class of functions such as algebraic and trigonometric polynomials as well as multivariate splines. The idea of studying such linear operators may be traced back to Kantorovich's modification of Bernstein polynomial operators. Various investigations of other positive linear operators involving both summation and integration in the one-variable setting, and particularly those that study the inverse problems and saturation properties, can be found in numerous papers (cf., for instance, [1, 2, 4, 5, 6, 10–14]).

To be more specific, we mention the most familiar (univariate) positive linear operator B_n introduced by Bernstein, namely

$$B_n(f)(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{nk}(x), \qquad f\in C[0,1],$$

where $b_{nk}(x)=\binom{n}{k} x^k(1-x)^{n-k}$. For functions f which are not necessarily continuous, Kantorovich introduced the positive linear smoothing operator

$$K_n(f)(x)=\sum_{k=0}^{n}\left(\int_0^1 (n+1)\,\chi_n\left(t-\frac{k}{n+1}\right) f(t)\,dt\right) b_{nk}(x),$$

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where χ_n is the characteristic function of the interval $(0, 1/(n+1)]$. It is intuitively clear that the sequence $\{(n+1)\chi_n\}$ should be replaceable by an arbitrary sequence of not necessarily positive approximate identity, as long as it is uniformly bounded in $L_1[0, 1]$. In this paper, this statement is rigorously verified, and more generally, a necessary and some sufficient conditions on this smoothing kernel will be given to guarantee the approximation property of the linear smoothing operators using both summation and integration, and this will be done in arbitrary dimension.

We will use the standard multivariate notation: $t = (t_1, ..., t_s) = (t(1), ..., t(s))$ and $dt = dt_1 \cdots dt_s$, etc., where $t_i = t(i)$ denotes the ith component of t. Moreover, the following notations will be used throughout this paper:

D: a bounded region, not necessarily open or closed, in $\mathbb{R}^s$.

$\bar{D}$: the closure of D.

$PC(D)$: the set of all piecewise continuous functions on D.

$L_p(D)$, $1 \leqslant p < \infty$: the usual Banach space with norm $\|\cdot\|_p$.

F: a closed subset of D.

$\|\cdot\|_F$: the supremum norm over F.

$\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_0$: the set of all integers, the positive ones, and the non-negative ones, respectively.

Δ_n: a subset of $\mathbb{Z}^s$ with cardinality n.

$\{\chi_{nk}\}$: a family of sets in D, $k \in \Delta_n$, $n \in \mathbb{Z}_+$.

λ_{nk}: real-valued functions in $L_1(D)$.

ω_{nk}: real-valued functions in $L_1(D)$ with unit mean.

Now consider the kernel

$$S_n(x, t) = \sum_{k \in \Delta_n} \lambda_{nk}(x)\, \omega_{nk}(t), \qquad x, t \in D,$$

of the linear operator Λ_n defined by

$$\Lambda_n(f)(x) = \int_D S_n(x, t)\, f(t)\, dt$$

$$= \sum_{k \in \Delta_n} \lambda_{nk}(x) \int_D \omega_{nk}(t)\, f(t)\, dt,$$

where $f: D \to \mathbb{R}$ belongs to a certain given class of functions.

We will investigate general conditions on the kernels S_n of Λ_n so that

$$\Lambda_n(f) \to f$$

as $n \to \infty$ for all f in $L_p(D)$, $1 \leqslant p < \infty$, or $C(F)$ where F is any closed subset of D as well as discuss the degree of approximation of f by $\Lambda_n(f)$ in the norms $\|\cdot\|_p$ and $\|\cdot\|_F$, respectively.

Our main results will be discussed in Sections 3 and 4 where some general convergence theorems and quantitative estimates are obtained. However, in our study of quantitative approximation we have to assume $\{\Lambda_n\}$ to be positive linear operators, i.e., both $\{\lambda_{nk}\}$ and $\{\omega_{nk}\}$ are nonnegative kernels. This paper concludes with a number of explicit kernels that can be used to generate various smoothing approximation operators.

2. Basic Assumptions

For convenience, we will always call $\{\lambda_{nk}\}$ and $\{\omega_{nk}\}$ the summation and smoothing kernels of Λ_n, respectively. In general, the integrable unit-mean functions ω_{nk} may not be positive. However, we assume that they are uniformly bounded in $L_1(D)$, namely

$$\int_D |\omega_{nk}(t)|\, dt \leqslant M < \infty, \qquad k \in \Delta_n \quad \text{and} \quad n \in \mathbb{Z}_+,$$

where $M \geqslant 1$ is some positive constant. This is a basic assumption on the smoothing kernel. Of course, this condition is always satisfied if $\omega_{nk} \geqslant 0$ with $M = 1$, since ω_{nk} is always assumed to have unit mean.

If we allow ω_{nk} to be replaced by a generalized function, say $\omega_{nk}(t) = \delta(t - x_{nk})$, where δ is the delta distribution at 0, then the operator Λ_n must be replaced by a linear operator L_n on $C(D)$, namely

$$L_n(f)(x) = \sum_{k \in \Delta_n} f(x_{nk})\, \lambda_{nk}(x), \qquad f \in C(D).$$

This may be called the summation operator associated with Λ_n and $\{x_{nk}\}$, the family of sample points associated with both L_n and Λ_n simultaneously. Thus, it may be said that operators Λ_n are completely determined by $(D; \{\lambda_{nk}\}, \{\omega_{nk}\}, \{x_{nk}\})$.

Moreover, throughout this paper we will always assume that each λ_{nk} is nonnegative on D, so that L_n is a positive linear operator. Certainly, in general the sequence of operators $\{\Lambda_n\}$ cannot enjoy any kind of convergence properties unless $\{L_n\}$ does. Let us now introduce a pair of Korovkin conditions for $\{L_n\}$ with $s+2$ test functions,

$$\begin{aligned} \varphi_0(x) &= 1 \\ \varphi_i(x) &= x(i), \qquad i = 1, ..., s \\ \varphi_{s+1}(x) &= (x(1))^2 + \cdots + (x(s))^2, \end{aligned} \tag{1}$$

as follows:

($\mathbb{K}_1$): $\{\lambda_{nk}\} \subset PC(D)$ and $\{x_{nk}\} \subset D$ satisfy

$$\max_{x \in F} \left| \sum_{k \in \Delta_n} \lambda_{nk}(x)\, \varphi_i(x_{nk}) - \varphi_i(x) \right| \to 0$$

for any closed set F in D, $i = 0, ..., s+1$.

($\mathbb{K}_2$): $\{\lambda_{nk}\} \subset L_p(D)$ and $\{x_{nk}\} \subset D$ satisfy

$$\left\| \sum_{k \in \Delta_n} \lambda_{nk}(\cdot)\, \varphi_i(x_{nk}) - \varphi_i(\cdot) \right\|_p \to 0$$

for $i = 0, ..., s+1$, where $1 \leqslant p < \infty$.

Clearly, these are basic conditions for the convergence of $\{L_n(f)\}$ to f in $C(F)$ and $L_p(D)$, respectively. Accordingly, we will denote the class of all kernels $\{\lambda_{nk}\}$ that satisfy Condition $\mathbb{K}_i$, for a given family of points $\{x_{nk}\}$ in D, by $[\mathbb{K}_i]$, $i = 1, 2$.

In the next section we will determine very general conditions on $\{\omega_{nk}\}$ so that $\{\Lambda_n\}$ will possess the almost uniform convergence property for $C(D)$, that is, uniform convergence on all closed subsets of D, as well as $L_p(D)$ convergence.

3. Convergence Theorems

In what follows we will always assume, without loss of generality, that D is a bounded region contained in $\mathbb{R}^s_+ = (0, \infty)^s$, so that every point $x \in D$ has positive components $x(i)$, $i = 1, ..., s$. By $|x|$, we mean the Euclidean length of x. Of course, the basic condition $\|\omega_{nk}\|_1 \leqslant M$ is always assumed for the unit-mean smoothing kernel $\{\omega_{nk}\}$.

Theorem 1. *Let $\lambda_{nk} \in [\mathbb{K}_1]$. If the family of kernels $\{\omega_{nk}\} \subset L_1(D)$ satisfies the condition*

$$\lim_{n \to \infty} \max_{k \in \Delta_n} \int_D |\omega_{nk}(t)|\, |t - x_{nk}|^2\, dt = 0, \qquad (\mathbb{H}_1)$$

then the sequence of operators $\{\Lambda_n\}$ satisfies the almost uniform convergence property on $C(D)$ in the sense that for any $f \in C(D)$, $\{\Lambda_n(f)\}$ converges to f uniformly on every closed subset of D. Furthermore, if, in addition, $\omega_{nk} \geqslant 0$ and $\{\lambda_{nk}\}$ satisfy

$$\|\lambda_{nk}\|_F \geqslant c > 0 \qquad (2)$$

for all $k \in \Delta_n$ and all sufficiently large $n \in Z_+$, where F is some closed set in D and c some constant, then the condition $(\mathbb{H}_1)$ is also necessary.

Proof. (i) *Sufficiency.* Let $f \in C(D)$ and F be any closed subset of D. Then for any $x \in F$, regardless of ω_{nk} being nonnegative or not, we have

$$|(\Lambda_n(f)-f)(x)| \leqslant |\Lambda(f)(x) - f(x)\,\Lambda_n(1)(x)|$$
$$+ |f(x)\,\Lambda_n(1)(x) - f(x)| =: J_1 + J_2,$$

say. Since $\Lambda_n(1)(x) = \sum_{k \in \Delta_n} \lambda_{nk}(x)\,\varphi_0(x_{nk}) \to \varphi_0(x) = 1$ uniformly on F, J_2 is uniformly small on F for large values of n. To estimate J_1, we apply the Chebyshev-type and Cauchy inequalities, namely

$$J_1 = \left| \sum_{k \in \Delta_n} \lambda_{nk}(x) \int_D \omega_{nk}(t)[f(t) - f(x)]\,dt \right|$$
$$\leqslant \sum_{k \in \Delta_n} \lambda_{nk}(x) \int_D |\omega_{nk}(t)| \left[\varepsilon + \frac{2\|f\|_F}{\delta^2} |t-x|^2 \right] dt$$
$$\leqslant \left(\sum_{k \in \Delta} \lambda_{nk}(x) \right) \left[\varepsilon M + \frac{4\|f\|_F}{\delta^2} \max_{k \in \Delta_n} \int_D |\omega_{nk}(t)|\,|t-x_{nk}|^2\,dt \right]$$
$$+ \frac{4\|f\|_F}{\delta^2} M \sum_{k \in \Delta_n} \lambda_{nk}(x)\,|x - x_{nk}|^2,$$

where $\varepsilon > 0$ is arbitrary and $\delta = \delta(\varepsilon) > 0$. Hence, by using both Conditions $(\mathbb{H}_1)$ and $(\mathbb{K}_1)$, we also have $J_1 \to 0$ uniformly on F as $n \to \infty$.

(ii) *Necessity.* Let $\omega_{nk} \geqslant 0$ for all $k \in \Delta_n$ and all n. Then $\{\Lambda_n\}$ is a sequence of positive linear operators.

We first verify the condition

$$\|\Lambda_n(|\cdot - x_{nk}|^2)\|_F \to 0, \tag{3}$$

where F is any closed set in D. Indeed, for each fixed i, $1 \leqslant i \leqslant s$, define

$$\bar{\Lambda}_n(f)(x) = \sum_{k \in \Delta_n} \bar{\lambda}_{nk}(x) \int_D \omega_{nk}(t)\,f(t)\,dt,$$

where $\bar{\lambda}_{nk}(x) =: \lambda_{nk}(x)\,\varphi_i(x_{nk})/\varphi_i(x)$ which, by using the classical Korovkin theorem, can easily be shown to be in $[\mathbb{K}_1]$. Hence, for each i, $1 \leqslant i \leqslant s$,

$$\sum_{k \in \Delta_n} \lambda_{nk}(x)\,\varphi_i(x_{nk}) \int_D \omega_{nk}(t)\,\varphi_i(t)\,dt$$
$$= \varphi_i(x) \sum_{k \in \Delta_n} \bar{\lambda}_{nk}(x) \int_D \omega_{nk}(t)\,\varphi_i(t)\,dt$$
$$= \varphi_i(x)\,\bar{\Lambda}_n(\varphi_i)(x) \to \varphi_i^2(x)$$

uniformly on F, and this, in turn, implies that

$$\sum_{k\in\Delta_n}\lambda_{nk}(x)\int_D\omega_{nk}(t)\,|t-x_{nk}|^2\,dt$$

$$=\sum_{k\in\Delta_n}\lambda_{nk}(x)\int_D\omega_{nk}(t)\,\varphi_{s+1}(t)\,dt+\sum_{k\in\Delta_n}\lambda_{nk}(x)\,\varphi_{s+1}(x_{nk})$$

$$-2\sum_{i=1}^{s}\sum_{k\in\Delta_n}\lambda_{nk}(x)\,\varphi_i(x_{nk})\int_D\omega_{nk}(t)\,\varphi_i(t)\,dt$$

$$\to\varphi_{s+1}(x)+\varphi_{s+1}(x)-2\sum_{i=1}^{s}\varphi_i^2(x)=0$$

uniformly on F. This proves (3).

Suppose now (2) is satisfied, but Condition $(\mathbb{H}_1)$ is not. Then there exists a subsequence $n_j\to\infty$ such that

$$\max_{k\in\Delta_{n_j}}\int_D\omega_{n_jk}(t)\,|t-x_{nk}|^2\,dt\geqslant h>0$$

for some constant h and all n_j. For each n_j, choose $k_j=k(n_j)\in\Delta_{n_j}$ such that

$$\int_D\omega_{n_jk_j}(t)\,|t-x_{n_jk_j}|^2\,dt=\max_{k\in\Delta_{n_j}}\int_D\omega_{n_jk}(t)\,|t-x_{n_jk}|^2\,dt.$$

Hence, by (2), we have

$$\|\Lambda_{n_j}(|\cdot-x_{nk}|^2)\|_F$$

$$=\sup_{x\in F}\sum_{k\in\Delta_j}\lambda_{n_jk}(x)\int_D\omega_{n_jk}(t)\,|t-x_{n_jk}|^2\,dt$$

$$\geqslant\|\lambda_{n_jk_j}\|_F\int_D\omega_{n_jk_j}(t)\,|t-x_{n_jk_j}|^2\,dt$$

$$\geqslant ch>0,$$

contradicting (3). This completes the proof of the theorem.

We will denote, as before, the class of all kernels $\{\omega_{nk}\}$ that satisfy Condition $(\mathbb{H}_1)$ by $[\mathbb{H}_1]$.

It is clear that a stronger but somewhat more applicable condition than $(\mathbb{H}_1)$ can be stated as follows:

$$\text{For any }\delta>0,\qquad\max_{k\in\Delta_n}\sup_{x\in D}\{|\omega_{nk}(x)|:|x-x_{nk}|\geqslant\delta\}\to 0.\tag{$\mathbb{H}_1'$}$$

Consequently, the following results are obtained.

COROLLARY 1. *Let* $\{\omega_{nk}\} \subset L_1(D)$ *satisfy Condition* $(\mathbb{H}_1')$. *Then for any* $f \in C(D)$ *and any closed subset* $F \subset D$, $\|A_n(f) - f\|_F \to 0$.

COROLLARY 2. *Let* $\{\omega_{nk}\} \subset L_1(D)$ *satisfy* $(\mathbb{H}_1)$ *or* $(\mathbb{H}_1')$. *Then for any* $f \in L_1(D)$, $A_n(f)(x) \to f(x)$ *at any point* x *where* f *is continuous.*

Next, we will consider the special but very important case where the smoothing kernels $\{\omega_{nk}\}$ are generated by a single function. Let $\omega \in L_1(D)$ with

$$\int_D \omega(t)\, dt = 1$$

such that $\operatorname{supp}(\omega)$ contains the origin in its interior. Here, as usual, $\operatorname{supp}(\omega)$ denotes the closure of the support ω. Extend the domain of the definition of ω to all $\mathbb{R}^s$ by setting it to be 0 outside D. Now, let $\{\lambda_{nk}\} \subset [\mathbb{K}_1]$ with respect to the set $\{x_{nk}\}$, $k \in \Delta_n$. We may pick any family $\{x^*_{nk}\}$, $k \in \Delta_n$, satisfying

$$\max_{k \in \Delta_n} |x^*_{nk} - x_{nk}| \to 0, \tag{4}$$

and define

$$\omega_{nk}(x) = n_1 \cdots n_s \omega(n_1(x - x^*_{nk})_1, \ldots, n_s(x - x^*_{nk})_s), \tag{5}$$

where for each $y \in \mathbb{R}^s$, $y_i = y(i)$, denotes the ith component of y.

Remark 1. By using (4), it is easy to verify that $\{\omega_{nk}\}$ defined by (5) is in $[\mathbb{H}_1]$. Hence, the linear summation-integral operator with kernel $\{\lambda_{nk}(x)\omega_{nk}(t)\}$ satisfies the almost uniform convergence property on $C(D)$ in the sense that $\|A_n(f) - f\|_F \to 0$ for all $f \in C(D)$ and any closed set $F \subset D$.

Remark 2. It is worth noting that Condition $(\mathbb{H}_1)$ may be viewed as a Korovkin condition for integral operators. In addition, in most examples, both $\{\lambda_{nk}\}$ and $(\omega_{nk}\}$ take on their maximum values at x_{nk}.

For L_p convergence, we have the following result.

THEOREM 2. *Let* $1 \leqslant p < \infty$ *and let* $\{\lambda_{nk}\}$, $\{\omega_{nk}\}$ *be in* $PC(D)$ *satisfying* $(\mathbb{K}_1)$, $(\mathbb{H}_1)$, *respectively. Suppose that, in addition, the following conditions are satisfied*:

(A) $$\max_{k \in \Delta_n} \int_D \lambda_{nk}(x)\, dx = O\left(\frac{1}{n}\right),$$

(B) $$\sup_{x \in D} \frac{1}{n} \sum_{k \in \Delta_n} |\omega_{nk}(x)| = O(1).$$

Then for every $f \in L_p(D)$, $\|A_n(f) - f\|_p \to 0$.

Proof. Recall that the kernel for Λ_n, considered as an integral operator, is

$$S_n(x, t) = \sum_{k \in \Delta_n} \lambda_{nk}(x)\, \omega_{nk}(t).$$

By Condition $(\mathbb{K}_1)$, we have

$$\int_D S_n(x, t)\, dt = \sum_{k \in \Delta_n} \lambda_{nk}(x) \to 1 \tag{6}$$

almost uniformly on D. Fix any $x \in D$ and any $\delta > 0$. Then

$$\begin{aligned}
\delta^2 \left| \int_{|t-x|>\delta} S_n(x, t)\, dt \right| &\leqslant \int_{|t-x|>\delta} |S_n(x, t)|\, |t-x|^2\, dt \\
&\leqslant \sum_{k \in \Delta_n} \lambda_{nk}(x) \int_D |\omega_{nk}(t)|\, |t-x|^2\, dt \\
&\leqslant 2 \sum_{k \in \Delta_n} \lambda_{nk}(x) \int_D |\omega_{nk}(t)|\, |t-x_{nk}|^2\, dt \\
&\quad + 2M \sum_{k \in \Delta_n} \lambda_{nk}(x)\, |x-x_{nk}|^2 \to 0.
\end{aligned}$$

Hence, for each $x \in D$ and $\delta > 0$,

$$\lim_{n \to \infty} \int_{|t-x|>\delta} S_n(x, t)\, dt = 0. \tag{7}$$

Using (6) and (7), we can now appeal to the multivariate Orlicz's theorem. To do so, it suffices to show the existence of an absolute constant C such that

$$\int_D |S_n(x, t)|\, dt < C, \qquad \int_D |S_n(x, t)|\, dx < C$$

on D for all n. The first one is obvious since $\|\omega_{nk}\|_1 \leqslant M$ and $\lambda_{nk} \geqslant 0$. The second bound is a consequence of the additional assumptions (A) and (B). Indeed, we have

$$\int_D |S_n(x, t)|\, dx \leqslant \frac{1}{n} \sum_{k \in \Delta_n} |\omega_{nk}(t)| \int_D n\lambda_{nk}(x)\, dx.$$

This completes the proof of the theorem.

Remark 3. In Theorem 2, we have assumed that $\{\omega_{nk}\} \subset PC(D)$ and Conditions (A) and (B) are satisfied. Although all the commonly used kernels have these properties, the assumptions are still somewhat unsatisfactory. In the next section, we will drop these assumptions when we specialize on positive operators Λ_n.

4. Quantitative Estimates

In order to apply the standard techniques to give quantitative estimates, we must specialize on positive linear operators; that is, we will assume that all ω_{nk} are nonnegative on D. The following notation will be used

$$\mu_n = \sup_{x \in D} \left(\sum_{k \in \Delta_n} \lambda_{nk} |x - x_{nk}|^2 \right)^{1/2}$$

$$\nu_n = \sup_{k \in \Delta_n} \left(\int_D \omega_{nk}(t) |t - x_{nk}|^2 \, dt \right)^{1/2}$$

$$\alpha_n = \max\{\mu_n, \nu_n\},$$

and the modulus of continuity for functions $f \in C(\bar{D})$ is defined to be

$$\omega_f(\delta) = \sup\{|f(x) - f(y)| : x, y \in \bar{D}, |x - y| \leqslant \delta\}.$$

The following result can be considered as a modified Shisha–Mond's theorem for Λ_n. Since its proof involves no difficulty, we do not include it here (cf. [8, 7]).

Theorem 3. *Suppose that* $\bar{D} = D$ *and* $\omega_{nk} \geqslant 0$. *Then for each* $f \in C(D)$ *and any* $\eta > 0$,

$$\|\Lambda_n(f) - f\|_D \leqslant \|f\|_D \, \|\Lambda_n(1) - 1\|_D + \{\|\Lambda_n(1)\|_D + 2(1 + \|\Lambda_n(1)\|_D / \eta^2)\} \, \omega_f(\eta \alpha_n).$$

In particular, if $L_n(1) \equiv 1$ *on* $\bar{D}$, *then*

$$\|\Lambda_n(f) - f\|_D \leqslant \left\{1 + \left(\frac{2}{\eta}\right)^2\right\} \omega_f(\eta \alpha_n).$$

To study the convergence in L_p, we will make use of a fundamental result due to Berens and DeVore [3]. As usual, denote by $W_p^r(\bar{D})$, $1 \leqslant p \leqslant \infty$, the Sobolev spaces with semi-norms

$$\|f\|_{r,p} = \max_{|k|_1 \leqslant r} \|D^k f\|_p$$

when $|k|_1 =: k_1 + \cdots + k_s$. We remark that, in addition, $W^2_\infty(\bar{D})$ is defined in the sense of Stein [9], and in this situation, Berens and DeVore [3] showed that

$$\|A_n(f) - f\|_p \leqslant C \|f\|_{2,\infty} \eta_{p,n}, \tag{8}$$

where C is dependent only on the dimension s and

$$\eta_{p,n} =: \max_{0 \leqslant i \leqslant s+1} \|A_n(\varphi_i) - \varphi_i\|_p. \tag{9}$$

Hence, we need an estimate on the sequence $\{\eta_{p,n}\}$. Of course, the Condition $(\mathbb{K}_2)$ must be satisfied in the first place. That is, we should have some idea on the rate of convergence of the following quantities to zero:

$$\delta_{i,n} =: \left\| \sum_{k \in \Delta_n} \lambda_{nk}(\cdot)\, \varphi_i(x_{nk}) - \varphi_i(\cdot) \right\|_p,$$

$i = 0, ..., s+1$. We set

$$\delta_n = \max_{1 \leqslant i \leqslant s} \delta_{i,n}$$

and will give estimates in terms of $\delta_{0,n}$, δ_n, and $\delta_{s+1,n}$. Next, the kernel $\{\omega_{nk}\}$, will be assumed to be nonnegative. It is clear that the following "Korovkin condition" is equivalent to the condition (H_1):

$$\delta_n^{**} =: \max_{k \in \Delta_n} \int_D \omega_{nk}(t)\, |t - x_{nk}|\, dt \to 0. \tag{H$_2$}$$

The quantity δ_n^{**} clearly dominates

$$\delta_n^* =: \max_{1 \leqslant i \leqslant s} \max_{k \in \Delta_n} \int_D \omega_{nk}\, |\varphi_i(t) - \varphi_i(x_{nk})|\, dt \tag{10}$$

and our estimate of $\eta_{p,n}$ will also be in terms of this smaller quantity δ_n^*. In the process of our estimation, the bounded sequence $\{\rho_n\}$ defined by

$$\rho_n =: \left\| \sum_{k \in \Delta_n} \lambda_{nk} \right\|_p$$

will appear, and we will set

$$\bar{c} = \sup_n \rho_n.$$

We have the following result.

THEOREM 4. *Let* $\{\lambda_{nk}\} \subset [\mathbb{K}_2] \cap L_p(D)$ *and* $\{\omega_{nk}\} \subset [\mathbb{H}_2] \cap L_q(D)$, $\omega_{nk} \geqslant 0$, *where* $1 \leqslant p < \infty$ *and* $(1/p) + (1/q) = 1$. *Then* $\|\Lambda_n(f) - f\|_p \to 0$ *for all* $f \in L_p(D)$. *In particular, if* $f \in W^2_\infty(\bar{D})$, *then*

$$\|\Lambda_n(f) - f\|_p \leqslant C\|f\|_{2,\infty}\beta_n, \tag{11}$$

where

$$\beta_n = \max\{\delta_{0,n}, \delta_n + \bar{c}\delta_n^*, \delta_{s+1,n} + c'\,\delta_n^*\} \tag{12}$$

with $c' = s\bar{c}\,\mathrm{diam}(D)$.

Proof. Since $W^2_\infty(\bar{D})$ is dense in $L_p(D)$, it is sufficient to prove (11). In view of (8), we simply have to verify $\eta_{p,n} \leqslant \beta_n$, where $\eta_{p,n}$ is defined in (9). Now,

$$\|\Lambda_n(\varphi_0) - \varphi_0\|_p = \left\| \sum_{k \in \Delta_n} \lambda_{nk}(\cdot) - 1 \right\|_p = \delta_{0,n}$$

and for $i = 1, ..., s$,

$$\begin{aligned}
\|\Lambda_n(\varphi_i) - \varphi_i\|_p &\leqslant \|\Lambda_n(\varphi_i) - L_n(\varphi_i)\|_p + \|L_n(\varphi_i) - \varphi_i\|_p \\
&\leqslant \left\{ \max_{k \in \Delta_n} \int_D \omega_{nk}(t)\,|\varphi_i(t) - \varphi_i(x_{nk})|\,dt \right\} \rho_n + \delta_{i,n} \\
&\leqslant \bar{c}\,\delta_n^* + \delta_n.
\end{aligned}$$

Finally, for $i = s+1$, we have

$$\begin{aligned}
&\|\Lambda_n(\varphi_{s+1}) - \varphi_{s+1}\|_p \\
&\quad\leqslant \left\| \sum_{k \in \Delta_n} \lambda_{nk}(\cdot) \int_D \omega_{nk}(t)\,\big|\,|t|^2 - |x_{nk}|^2\big|\,dt \right\|_p + \delta_{s+1,n} \\
&\quad\leqslant \delta_n^* \rho_n \cdot (s)(\mathrm{diam}\ D) + \delta_{s+1,n} \\
&\quad\leqslant c'\,\delta_n^* + \delta_{s+1,n}
\end{aligned}$$

as required. This completes the proof of the theorem.

Remark 4. If D is a closed set, then $(\mathbb{K}_2)$ is a weaker condition than $(\mathbb{K}_1)$. It is not clear, however, if $(\mathbb{H}_2)$ is a necessary condition on $\{\omega_{nk}\}$ for $L_p(D)$ convergence of $\{\Lambda_n(f)\}$ to f where $f \in L_p(D)$ when a suitable subclass of kernels $\{\lambda_{nk}\} \subset [\mathbb{K}_2] \cap L_p(D)$ is considered.

Remark 5. More generally, it is possible to associate $\{\lambda_{nk}\}$ and $\{\omega_{nk}\}$ with two different families of points in D, $\{x_{nk}\}$ and $\{x^*_{nk}\}$, say, as long as the sequence

$$\max_{k \in \Delta_n} |x_{nk} - x^*_{nk}|$$

tends to zero, and if quantitative estimates are desired, it must tend to zero at a suitable rate.

5. Examples

There are multivariate versions of the well known positive kernels of one variable. These kernels can be used for $\{\lambda_{nk}\}$. Of course, one way to construct ω_{nk} is to set $\omega_{nk} = c_{nk}\lambda_{nk}$ where

$$c_{nk} = \left(\int_D \lambda_{nk}(t)\, dt\right)^{-1}. \tag{13}$$

Quite often, we have $c_{nk} = c_n$ independent of k and in this situation we may substitute c_{nk} by an asymptotic value α_n of c_n unless a quantitative estimate is required. In the following, we present four examples for $\{\lambda_{nk}\}$, the first three on algebraic polynomials and the last on trigonometric polynomials. Of course, a different set of test functions is required for the latter case and the Korovkin conditions $(\mathbb{K}_i)$ and $(\mathbb{H}_i)$, $i = 1, 2$, have to be changed accordingly.

Example 1 (Bernstein Polynomials on a Simplex). Let $D = \{x = (x_1, ..., x_s): \sum_{i=1}^s x_i \leq 1, x_i \geq 0\}$ and $\Delta_n = \{k = (k_1, ..., k_s) \in \mathbb{Z}_0^s: \sum_{i=1}^s k_i \leq N\}$, where $n = \binom{N+s}{s} \sim N^s/s!$. Set

$$x_{nk} = \frac{k}{N} =: \left(\frac{k_1}{N}, ..., \frac{k_s}{N}\right)$$

and

$$\lambda_{nk}^{(1)}(x) = \binom{N}{k} x^k \left(1 - \sum_{i=1}^s x_i\right)^{N-k_1-\cdots-k_s}.$$

Here, the usual multivariate notation

$$\binom{N}{k} = \frac{N!}{k_1! \cdots k_s!\,(N-k_1-\cdots-k_s)!}, \qquad x^k = x_1^{k_1}\cdots x_s^{k_s}$$

is used. Note that the normalization constants c_{nk} defined by (13) and their asymptotic values α_n are

$$c_{nk} = \frac{(N+s)!}{N!} \sim N^s$$

for all k.

EXAMPLE 2 (Landau Polynomials on a Sphere).Let $D = \{x: |x| \leqslant \frac{1}{2}\}$ and $\Delta_n = \{k \in Z_0^s: |k| < N/2\}$ with cardinality n. We define $x_{nk} = k/n := (k_1/N, ..., k_s/N)$ and

$$\lambda_{nk}^{(2)}(x) = \left(\frac{1}{N\pi}\right)^{s/2} \left(1 - \left|x - \frac{k}{N}\right|^2\right)^N.$$

It may be shown that c_{nk} is independent of k, $c_{nk} \sim N^s$, and

$$n \sim \frac{\pi^{s/2}}{2^s \Gamma((s/2)+1)} N^s.$$

EXAMPLE 3 (Landau Polynomials on a Cube). Let $D = [-\frac{1}{2}, \frac{1}{2}]^s$ and $\Delta_n = \{k = (k_1, ..., k_s) \in Z_0^s: |k_i| < N/2, i = 1, ..., s\}$ with $n \sim 2^s (N/2)^s = N^s$. Set $x_{nk} = k/N := (k_1/N, ..., k_s/N)$ and

$$\lambda_{nk}^{(3)}(x) = \left(\frac{1}{sN\pi}\right)^{s/2} \left(1 - \frac{1}{2}\left|x - \frac{k}{N}\right|^2\right)^N.$$

Here, the normalizing constant is also independent of k and $c_{nk} \sim N^s$.

EXAMPLE 4 (Multivariate Rappoport Polynomials on a Cube). Let $D = [0, 2\pi]$ and $\Delta_n = \{k \in Z_0^s: k_i \leqslant 2N, i = 1, ..., s\}$ with $n = (2N+1)^s$. Set $x_{nk} = (2\pi/(2N+1))k := \{(2\pi/(2N+1))\, k_1, ..., (2\pi/(2N+1))\, k_s\}$ and

$$\lambda_{nk}^{(4)}(x) = \left(\frac{\pi}{4sN}\right)^{s/2} \left[\frac{1}{s}\sum_{i=1}^{s} \cos^2\left(\frac{x_i - x_{nk}(i)}{2}\right)\right]^N,$$

where $x = (x_1, ..., x_s)$ and the normalizing constant is again independent of k and has asymptotic value $c_{nk} \sim (N/\pi)^s$.

That Conditions $(\mathbb{K}_1)$ and $(\mathbb{K}_2)$ are satisfied by the $\{\lambda_{nk}^{(i)}\}$ associated with the equally spaced $\{x_{nk}\}$ $\{k \in \Delta_n\}$ in Examples 1, 2, and 3 can be easily verified. Moreover, Conditions (A) and (B) in Theorem 2 as well as Condition $(\mathbb{H}_1')$ can also be justified for the $\lambda_{nk}^{(i)}$ and the corresponding kernels $\{\omega_{nk}^{(i)}\} = \{\alpha_n \lambda_{nk}^{(i)}\}$ with $\alpha_n \sim c_{nk}$.

In Example 4, it must be mentioned that the test functions in Conditions $(\mathbb{K}_1)$ and $(\mathbb{K}_2)$ should be replaced by the functions 1, $\cos x_1, ..., \cos x_s$, $\sin x_1, ..., \sin x_s$. Since

$$\int_{[0,\pi/4]^s} \left(\sum_{i=1}^{s} \cos^2 t_i\right)^N dt \sim s^N \frac{1}{2^s}\left(\frac{\pi s}{N}\right)^{s/2},$$

with $\omega_{nk}^{(4)} = (N/\pi)^s \lambda_{nk}^{(4)}$, Condition $(\mathbb{H}_1')$ can be verified.

Using the above examples, we can formulate several summation integral operators with kernels

$$S_n(x, t) = \sum_{k \in \Delta_n} \lambda_{nk}^{(i)}(x)\, \omega_{nk}^{(j)}(t)$$

for appropriate i and j. Of course, if the main purpose of $\{\omega_{nk}\}$ is to smooth the data function f, very simple but sufficiently smooth functions should be chosen. In particular, in view of Remark 1, any multivariate spline ω with compact support and unit mean can be used for this purpose.

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